Name of the Project Title :-

Mathematical Modelling in Population dynamics (Covid-19 case)

Chapter - I

Introduction

The Process of converting the real world problems into abstract mathematical problems, solving them and interpreting the solutions in the language of the real world is called Mathematical modelling.

Millions of People throughout the world are suffering from the Pandemic Covid-19 (Corona virus). This infectious disease spreaded in India when one or more infectives enter into the country (Population) from outside. It has been [redacted] found that there is a time gap between the receipt of infection and the appearance of symptoms in the Covid-19 case. After an individual gets infected by Corona disease, the symptoms of this Corona disease are manifested on the body of the Person after a gap of about two weeks. This time interval of two weeks is called incubation Period.

The total Population at any time may be divided into the following disjoint classes of individuals.

(i) Class of Susceptibles (x) :- This consists of those individuals who are not yet infected by the disease but are capable of developing or catching the disease and becoming infective at any time.

(ii) Class of infectives (y) :- This consists of those individuals who have already-been infected by the disease and are capable of transmitting the disease to susceptibles.

(iii) Class of removals (z) :- This consists of those individuals who have had the disease and are removed from the population. Removal from the population may take place in following three ways:-

(a.) The individual may be dead after suffering from corona.

(b.) The individual may be isolated or quarantined on appearance of symptoms of the disease.

(c.) The individual might have recovered from the disease corona and may acquire immunity to future attacks of corona.

It is quite natural that the density of class of susceptibles (x) will gradually reduce as more and more susceptibles get the disease through contacts between members of the classes (x) and (y). This will lead to increase in the density of (y). Similarly, the density of class (y) decreases gradually as corona spreads and more and more infectives are transferred to the class (z). Finally, a balance is established between classes (x), (y) and (z) as corona disease evolves and ultimately subsides.

In chapter 2, we shall formulate a mathematical model under above assumptions by taking the population of 'n' individuals; population being homogeneous.

Homogeneous group of n persons :- (Closed Population)

$x + y + z = n$; $x = x_0$; $y = y_0$; $z = 0$, when $t = 0$

Susceptibles, Infectives, removals (Initially; $x_0 + y_0 = n$) (Initial stage)

The no. of new infectives in time $(t, t+\Delta t) = \beta . x . y . \Delta t$ (β = Contact rate)

The no. of removals in time $(t, t+\Delta t) = \gamma . y . \Delta t$ (γ = removal rate)

$\therefore \quad \Delta x = -\beta . x . y . \Delta t$; $\Delta y = \beta x y \Delta t - \gamma . y . \Delta t$; $\Delta z = \gamma . y . \Delta t$

taking $\Delta t \to 0$; we get;

$$\frac{dx}{dt} = -\beta x . y \quad (1); \quad \frac{dy}{dt} = y(\beta x - \gamma) \quad (2); \quad \frac{dz}{dt} = \gamma . y \quad (3)$$

($\because$ New infectives come from susceptible class $\Rightarrow$ it equals Δx in x in time Δt
$\therefore -\beta x y . \Delta t = \Delta x$ because density of susceptible class decreases as a result of new infections)

$$\frac{\text{The rate of removal}}{\text{The rate of contact}} = \frac{\gamma}{\beta} = \rho \text{ (say)} = \text{relative removal rate}$$

= The rate at which persons are removed from the infected category to the rate at which they are added to the same (infected) category.

(1) $\Rightarrow \frac{dx}{dt} < 0 \ \forall \ t > 0 \Rightarrow x$ is m.d. function of time 't' (always)

(2) $\Rightarrow \frac{dy}{dt}$ at $t = 0 < 0$ if $\beta x - \gamma < 0 \Rightarrow x < \frac{\gamma}{\beta} \Rightarrow x_0 < \frac{\gamma}{\beta}$ ($\because$ at $t = 0$, $x = x_0$)

$\because x$ is m.d. function of time t $\Rightarrow x \leq x_0$ (x will go on decreasing as 't' increases)

$$\therefore \quad x \leq x_0 < \frac{\gamma}{\beta}$$

$\Rightarrow \frac{dy}{dt} < 0$ if $x_0 < \rho$. $\Rightarrow y$ is m.d. function of time t if $x_0 < \rho$ (not always)

$\Rightarrow$ epidemic (or pandemic) starts if $x_0 > \rho$

i.e. no epidemic will start if $x_0 < \rho$

This means threshold or critical no. of Susceptibles is ρ.
(ρ is a value which the initial number of susceptibles must exceed to enable the epidemic to spread)

* If infection is very-very small, then $y_0 \approx 0$

$$\Rightarrow x_0 \approx n$$

If $n > \rho$ then epidemic build-up will occur
and if $n < \rho$ then the initial trace of infection will be removed faster than it can be communicated to others.

If $y_\infty = 0 \Rightarrow$ infectives go to removals
then $\frac{z_\infty}{n}$ = measure of intensity of the epidemic

Hence; we shall find z.

(1) ÷ (3) $\Rightarrow$ $$\frac{dx}{dz} = -\frac{\beta}{\gamma} x = \frac{-x}{\rho}$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{x} = -\frac{1}{\rho}\int_0^z dz$$

$$\ln\left(\frac{x}{x_0}\right) = -\frac{z}{\rho}$$

$$\boxed{x = x_0 \cdot e^{-z/\rho}}$$

$$\geq x_0 \cdot e^{\frac{-n}{\rho}}$$

$$> 0$$

$(\because z \leq n)$

$$\frac{-z}{\rho} \geq \frac{-n}{\rho}$$

$\therefore \; x_\infty = \lim_{t\to\infty} x(t) > 0$

$\Rightarrow$ The ultimate density of susceptibles is non-zero.

$\Rightarrow$ The ultimate possibility of complete extinction of susceptible population is not possible.

$\Rightarrow$ Some susceptibles will escape the disease altogether

$\Rightarrow$ The spread of the disease will not stop altogether for the lack of susceptibles.

(3) $\Rightarrow \dfrac{dz}{dt} = \gamma \cdot y$

$$\frac{dz}{dt} = \gamma \cdot (n - x - z)$$

$$\frac{dz}{dt} = \gamma \left(n - z - x_0 \cdot e^{-z/\rho}\right)$$

$$\frac{dz}{dt} = \gamma \left\{ n - z - x_0 \left(1 - \frac{z}{\rho} + \frac{z^2}{\rho^2 \cdot 2}\right)\right\} \quad \left(\frac{z}{\rho} \text{ taken is very small}\right)$$

$$\frac{dz}{dt} = \gamma \left\{ n - z - x_0 + \frac{x_0 z}{\rho} - \frac{x_0 \cdot z^2}{2\rho^2} \right\}$$

$$\frac{dz}{dt} = \gamma \left\{ (n - x_0) + \left(\frac{x_0}{\rho} - 1\right) z - \frac{x_0 z^2}{2\rho^2} \right\}$$

$$\frac{dz}{dt} = \gamma \left[\gamma_0 - \frac{x_0}{2\rho^2} \left\{ z^2 - \frac{2\rho^2}{x_0}\left(\frac{x_0}{\rho} - 1\right) z + \left\{\frac{\rho^2}{x_0}\left(\frac{x_0}{\rho} - 1\right)\right\}^2 \right\} + \frac{x_0}{2\rho^2} \cdot \frac{\rho^4}{x_0^2}\left(\frac{x_0}{\rho} - 1\right)^2 \right]$$

$$\frac{dz}{dt} = \left[\gamma_0 - \frac{x_0}{2\rho^2} \left\{ z - \frac{\rho^2}{x_0}\left(\frac{x_0}{\rho} - 1\right)\right\}^2 + \frac{\rho^2}{2x_0}\left(\frac{x_0}{\rho} - 1\right)^2 \right]$$

(4)

On Solving;

$$Z = \frac{\rho^2}{x_0}\left[\left(\frac{x_0}{\rho}-1\right)+\alpha \tanh\left(\frac{1}{2}\alpha y t-\phi\right)\right] \quad (4)$$

which gives the approximate number of individuals removed by time 't'.

Imp:

We are therefore interested in the study of $\frac{dZ}{dt}$ which gives the rate of removal (PHD relates the no. of new removals each day or each week)

on diff (4) w.r.t z, the eqn of epidemic (Pandemic) curve is

$$\frac{dz}{dt} = \frac{y\alpha^2\rho^2}{2x_0}\operatorname{sech}^2\left(\frac{1}{2}\alpha y t-\phi\right)$$

∴ The epidemic curve for the ~~present~~ model will be obtained by plotting $\frac{dz}{dt}$ against t which is shown as

This shows that epidemic builds upto a peak and then dies away.

$$\therefore Z_\infty = \lim_{t\to\infty} Z = \frac{\rho^2}{x_0}\left(\frac{x_0}{\rho}-1+\alpha\right)$$

which give the ultimate size of the epidemic

$$\alpha = \left[\left(\frac{x_0}{\rho}-1\right)^2+\frac{2x_0y_0}{\rho^2}\right]^{1/2} = \text{constant}$$

$$\phi = \tanh^{-1}\left\{\frac{1}{\alpha}\left(\frac{x_0}{\rho}-1\right)\right\} = \text{constant}$$